Multiple Regression

General Matrix form for Multiple Regression

Rahul Singh
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What is Simple Linear Regression

re·gres·sion

noun

A statistical method used to **predict** the relationship between a dependent variable and one or more independent variables

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A statistical method used to **predict** the relationship between a dependent variable and one or more independent variables

in other words…

if we see some data (x, y) we can use linear regression to **predict** the y values for other values of x

What is Simple Linear Regression

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A statistical method used to **predict** the relationship between a dependent variable and one or more independent variables

$$y = f(x)$$

in other words…

if we see some data (x, y) we can use linear regression to **predict** the y values for other values of x

$\boxed{y \text{ is the dependent variable}}$

What is Simple Linear Regression

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in other words…

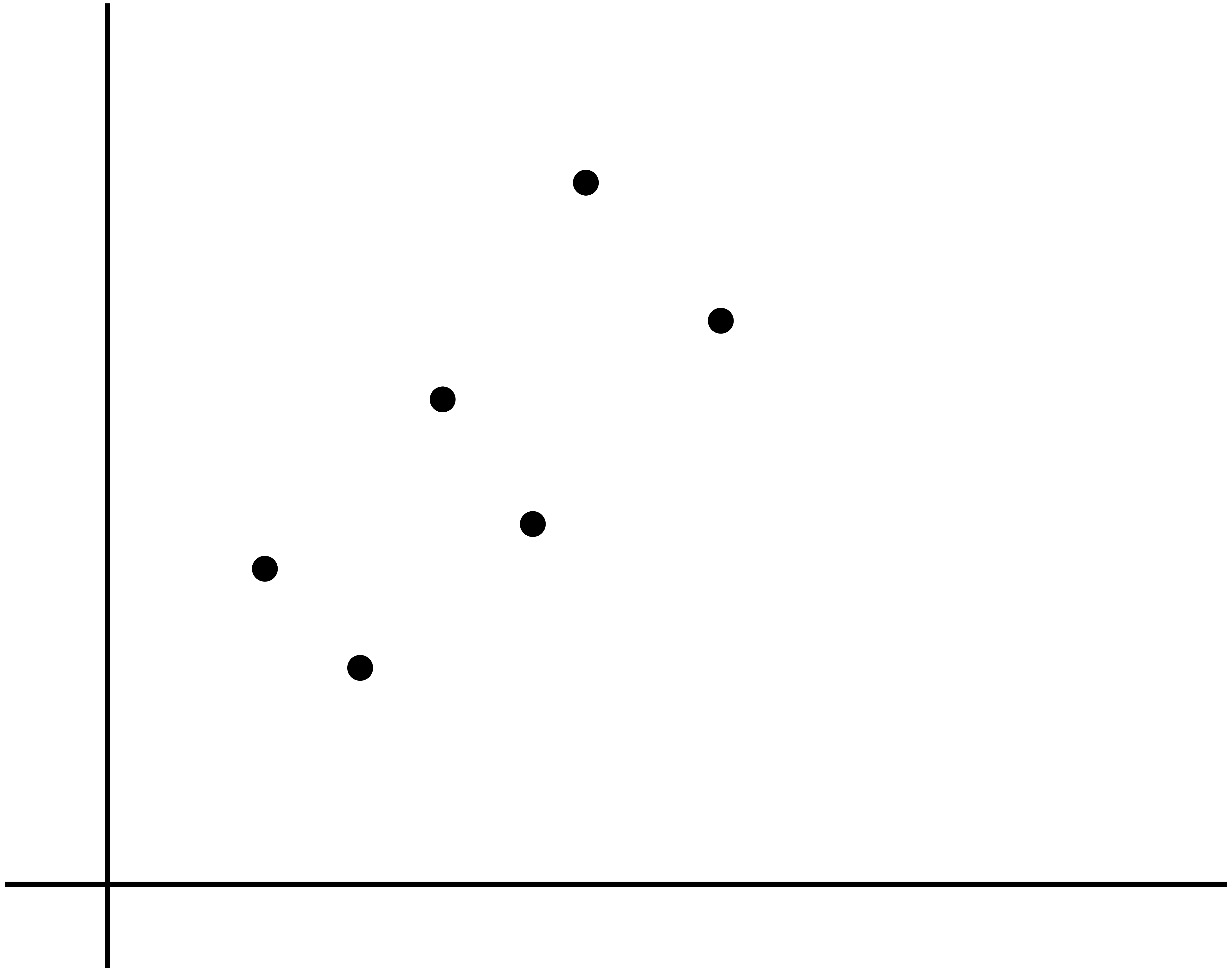
if we see some data (x, y) we can use linear regression to **predict** the y values for other values of x

x is the independent variable

y is the dependent variable

Simple Linear Regression

Fundamental Concept: Given a set of data (observations), find the values of β_0 and β_1 for the line that best fits the given data.



Data is synthetic and not plotted to scale

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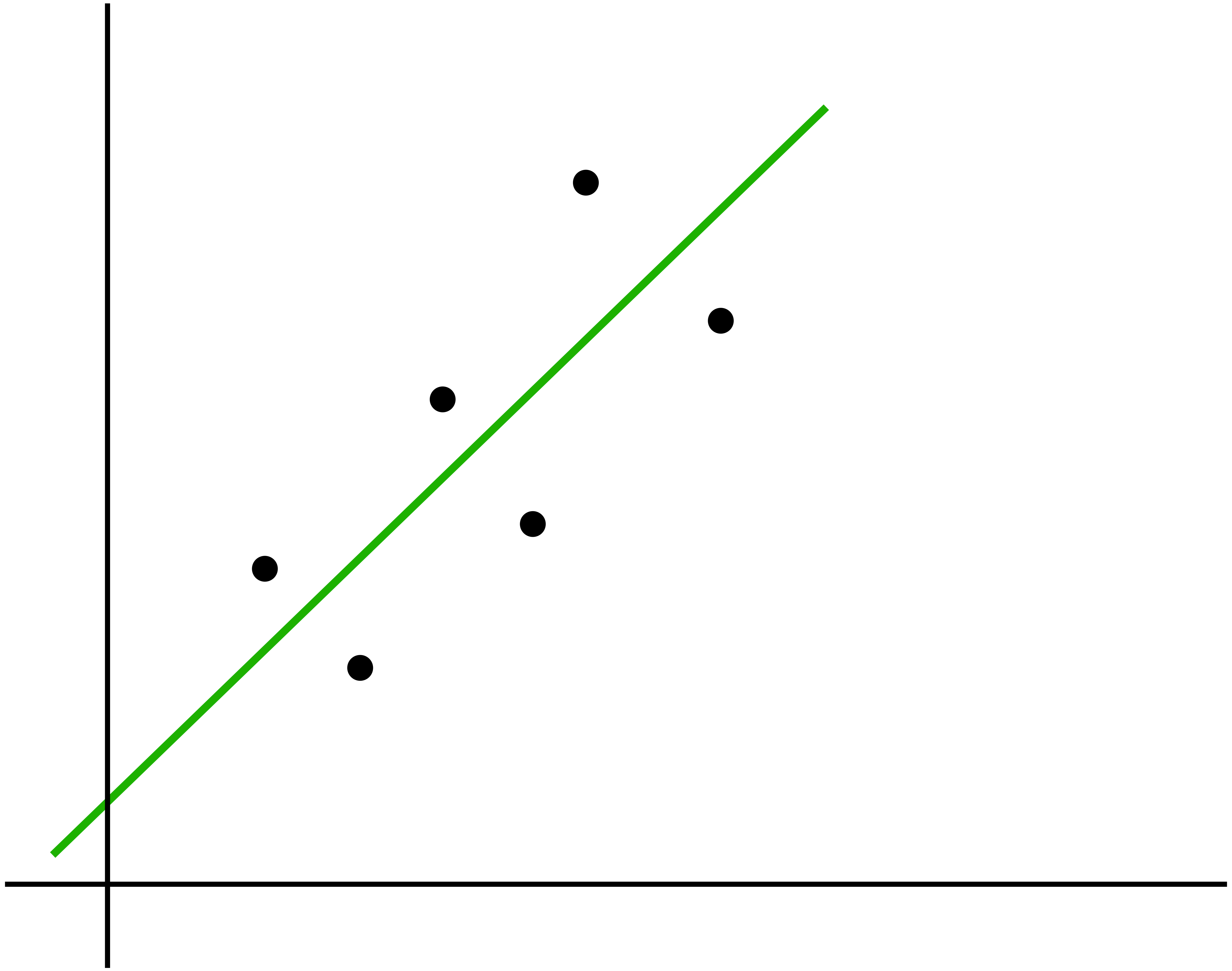

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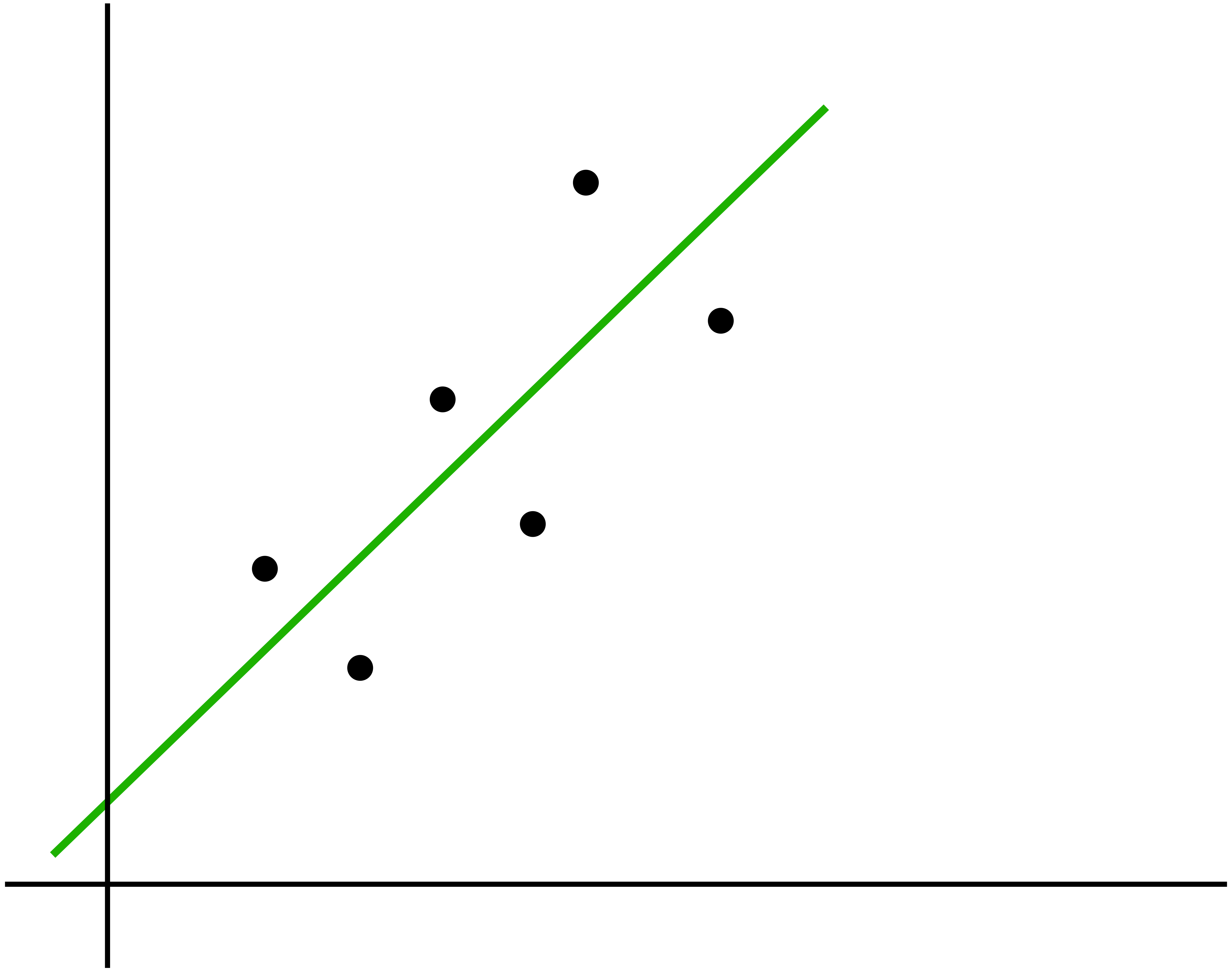
Fundamental Concept: Given a set of data (observations), find the values of β_0 and β_1 for the line that best fits the given data.

The line of best fit is $\hat{y} = \beta_0 + \beta_1 x$



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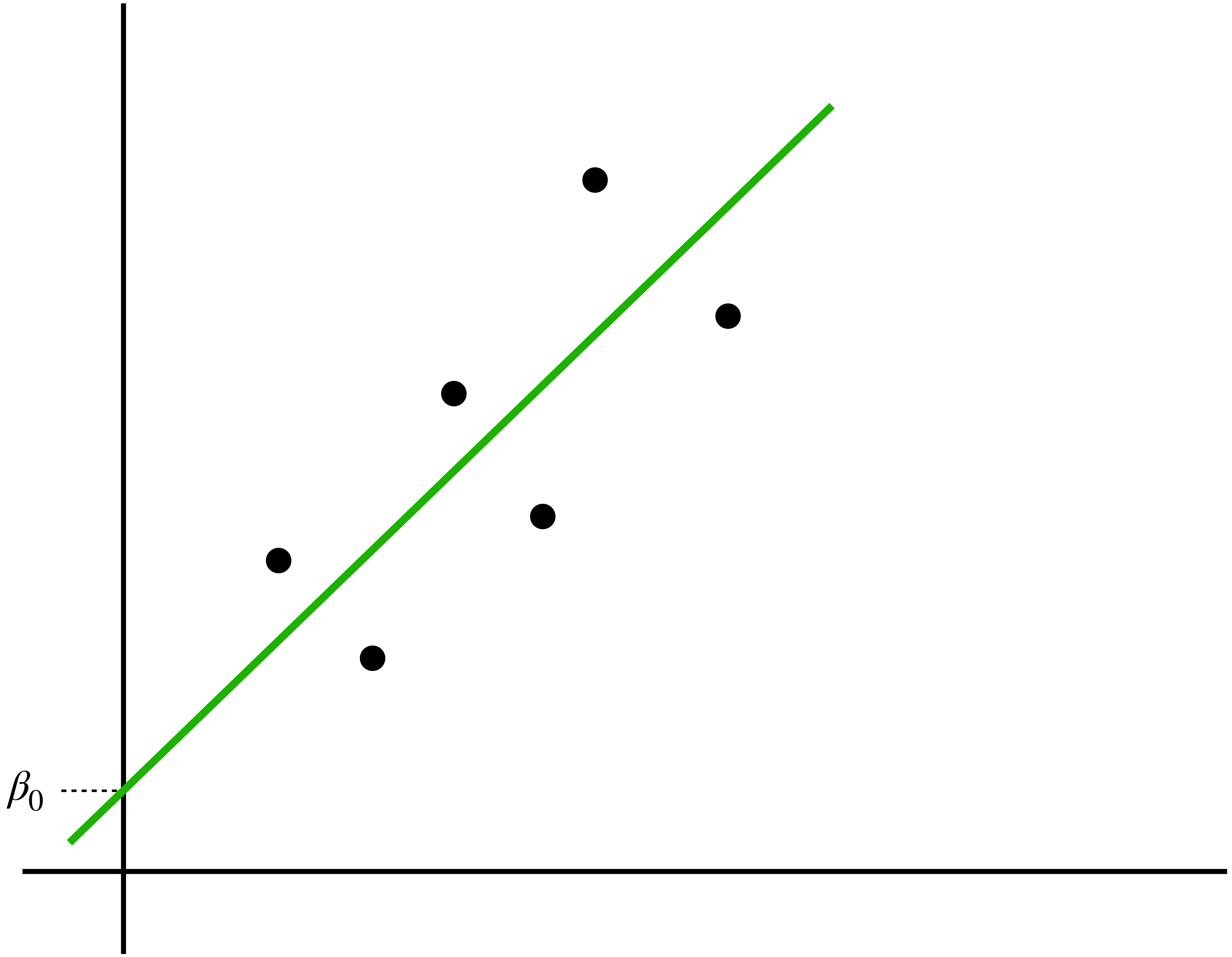

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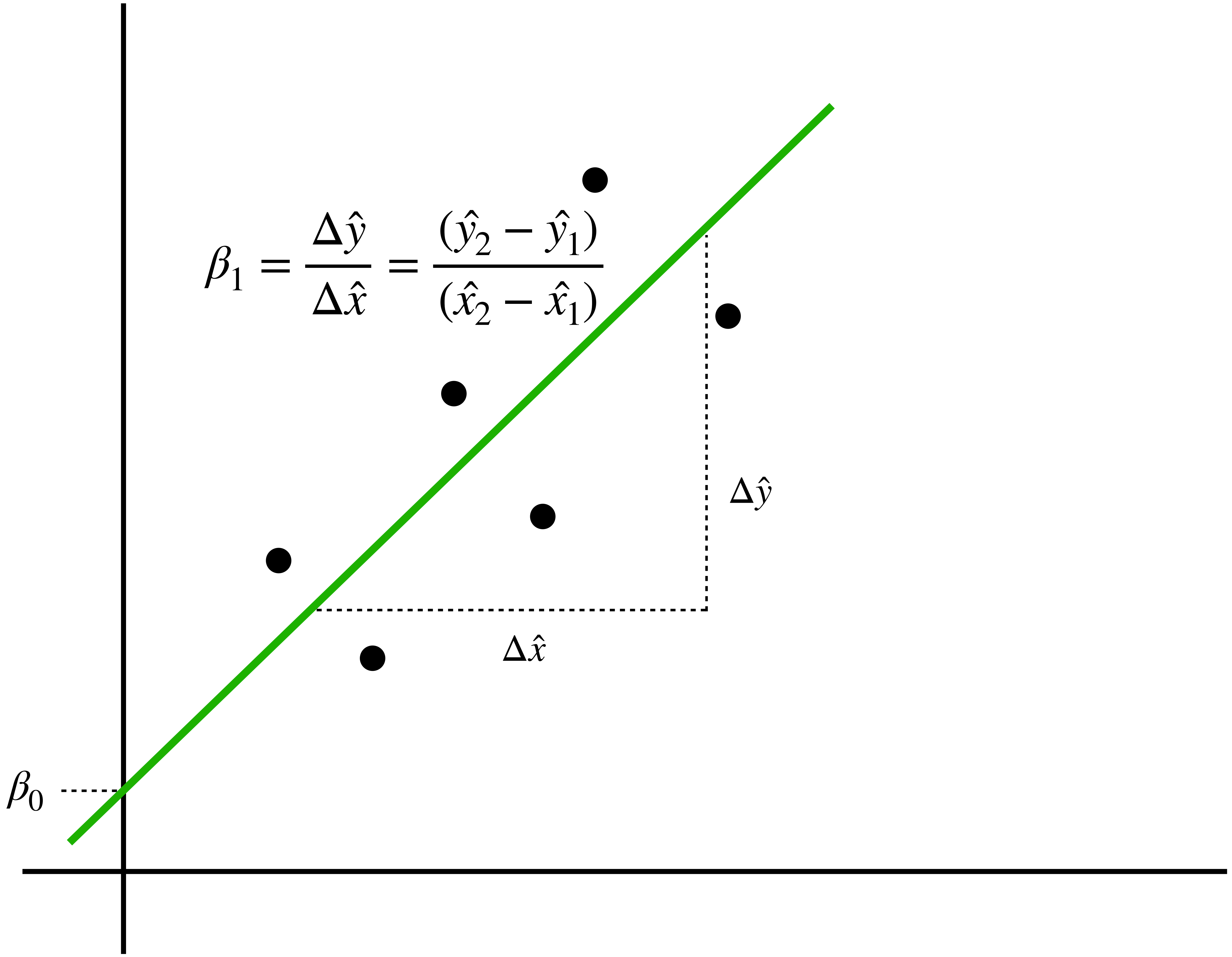
β_1 Is the slope of the line



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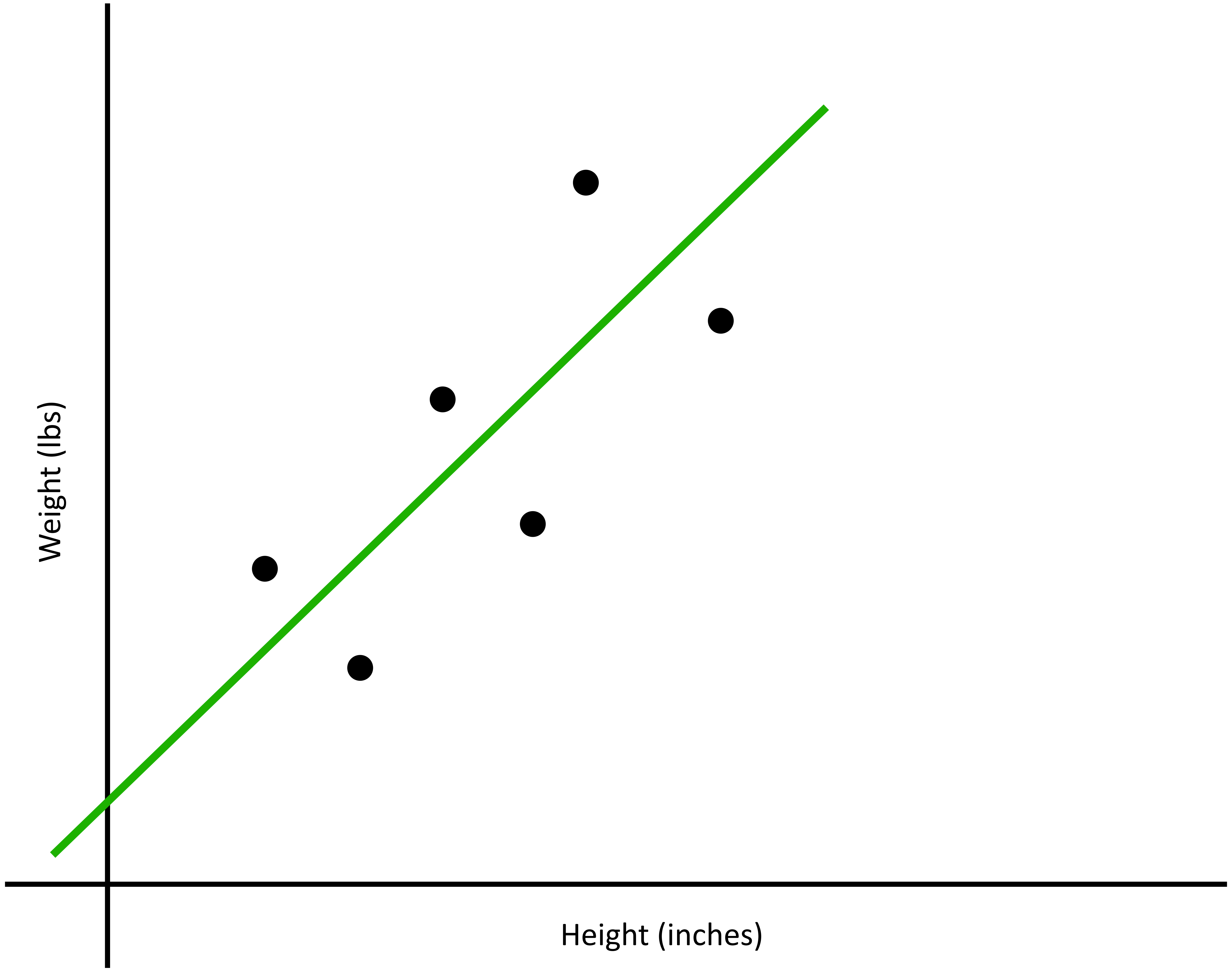
x is the independent variable

$\hat{y}$ is the dependent variable

Eg: Predict Weight ($\hat{y}$) given Height (x)



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Simple Linear Regression

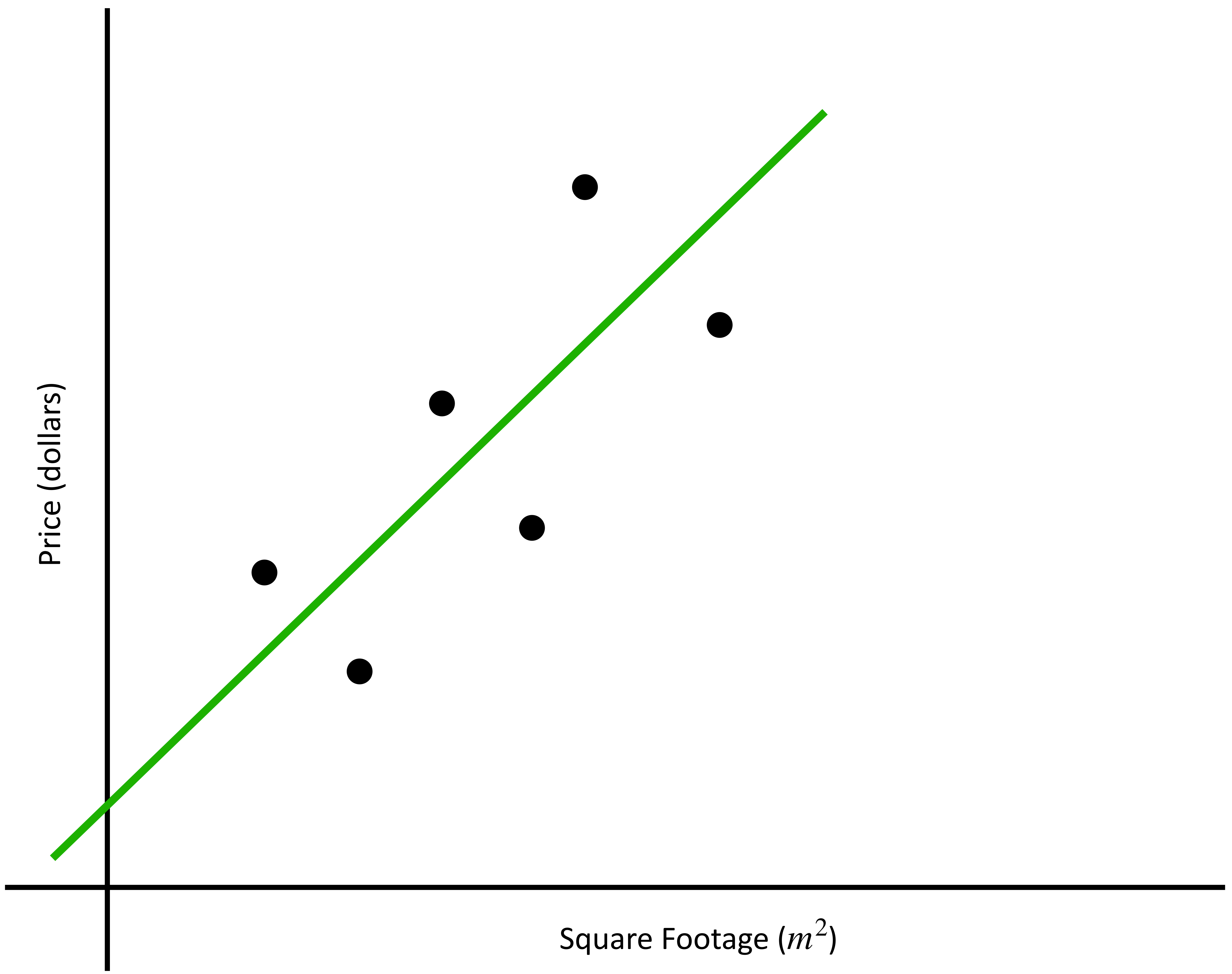
Fundamental Concept: Given a set of data (observations), find the values of β_0 and β_1 for the line that best fits the given data.

$$\hat{y} = \beta_0 + \beta_1 x$$

x represents square footage (m^2)

$\hat{y}$ represents price (dollars)

Eg: Predict Price of a house ($\hat{y}$) given Square Footage (x)



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Simple Linear Regression

Fundamental Concept: Given a set of data (observations), find the values of β_0 and β_1 for the line that best fits the given data.

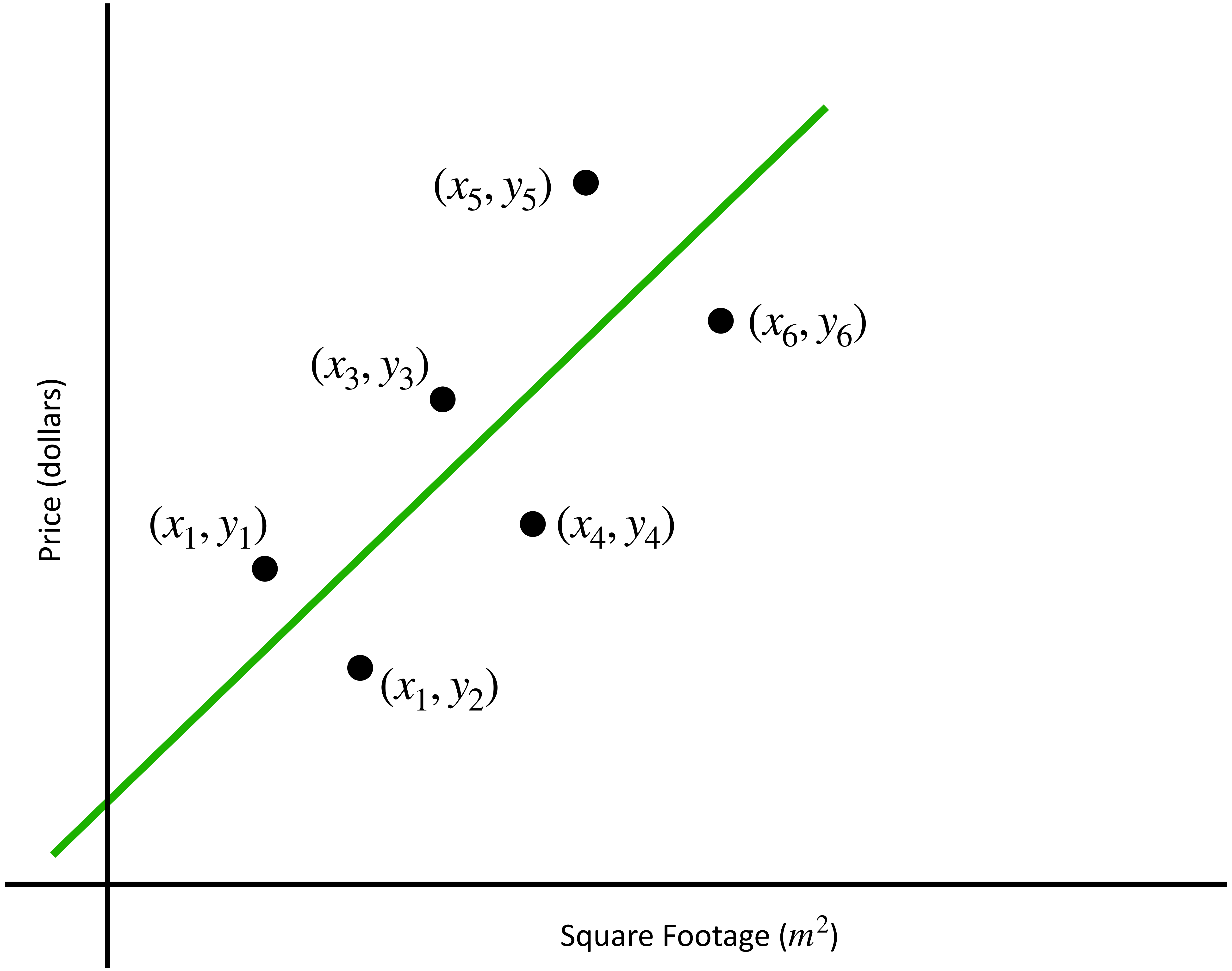
The line of best fit is $\hat{y} = \beta_0 + \beta_1 x$

For each point calculate the squared distance to the line. Divide that by the number of data points.

$$\frac{(y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2 + (y_3 - \hat{y}_3)^2 + (y_4 - \hat{y}_4)^2 + (y_5 - \hat{y}_5)^2 + (y_6 - \hat{y}_6)^2}{n}$$

This is the **Mean Squared Error (MSE)**

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$



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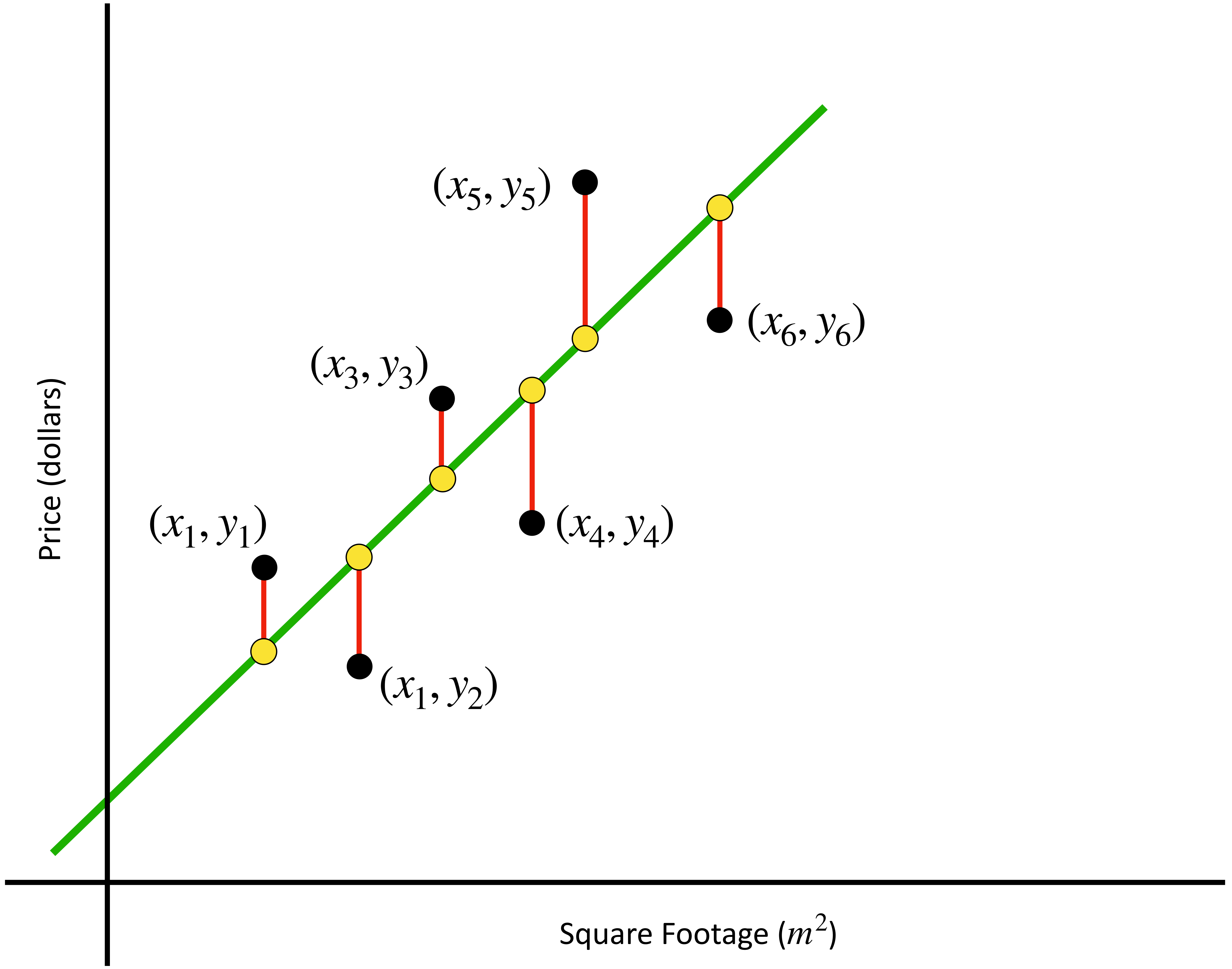
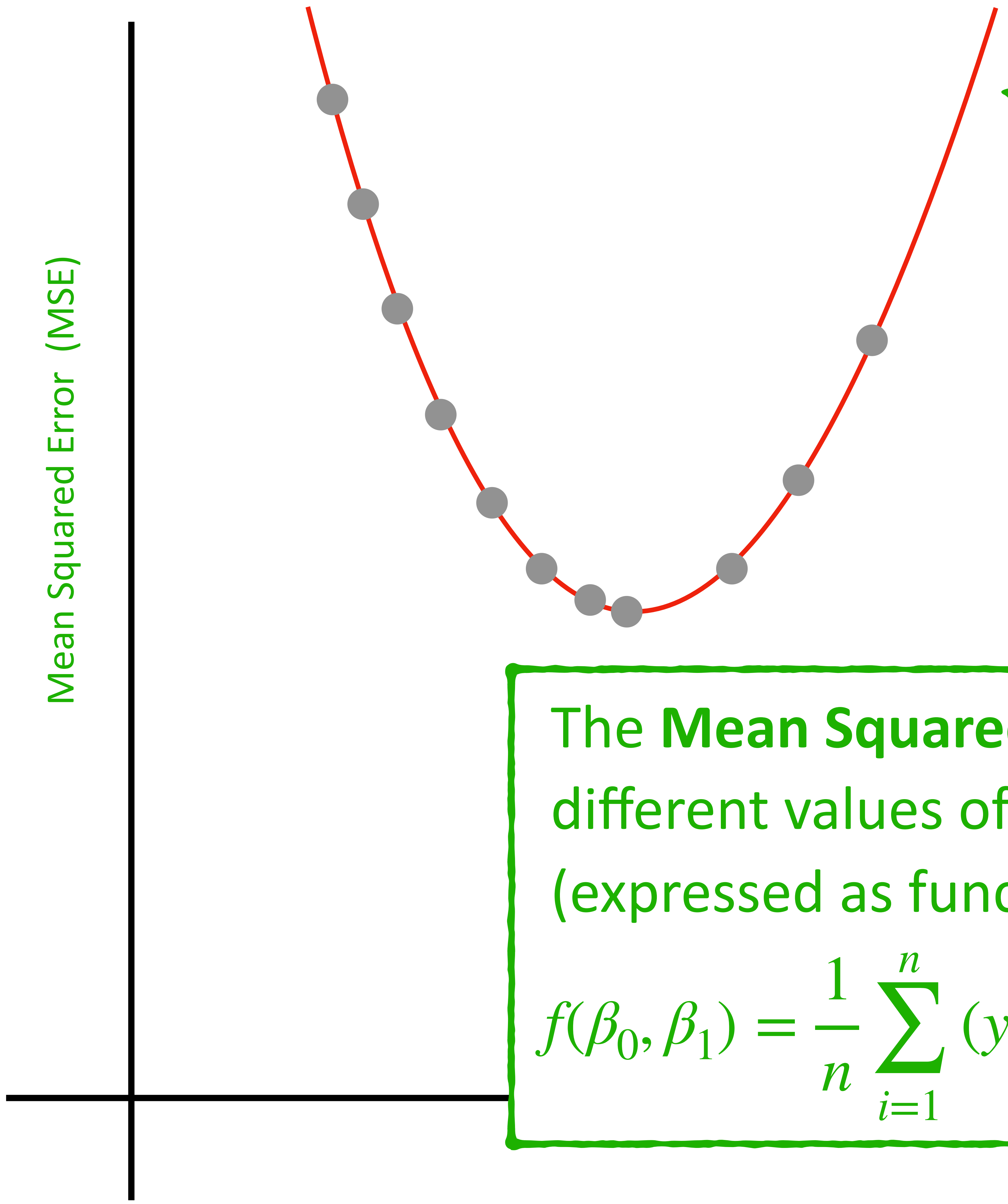


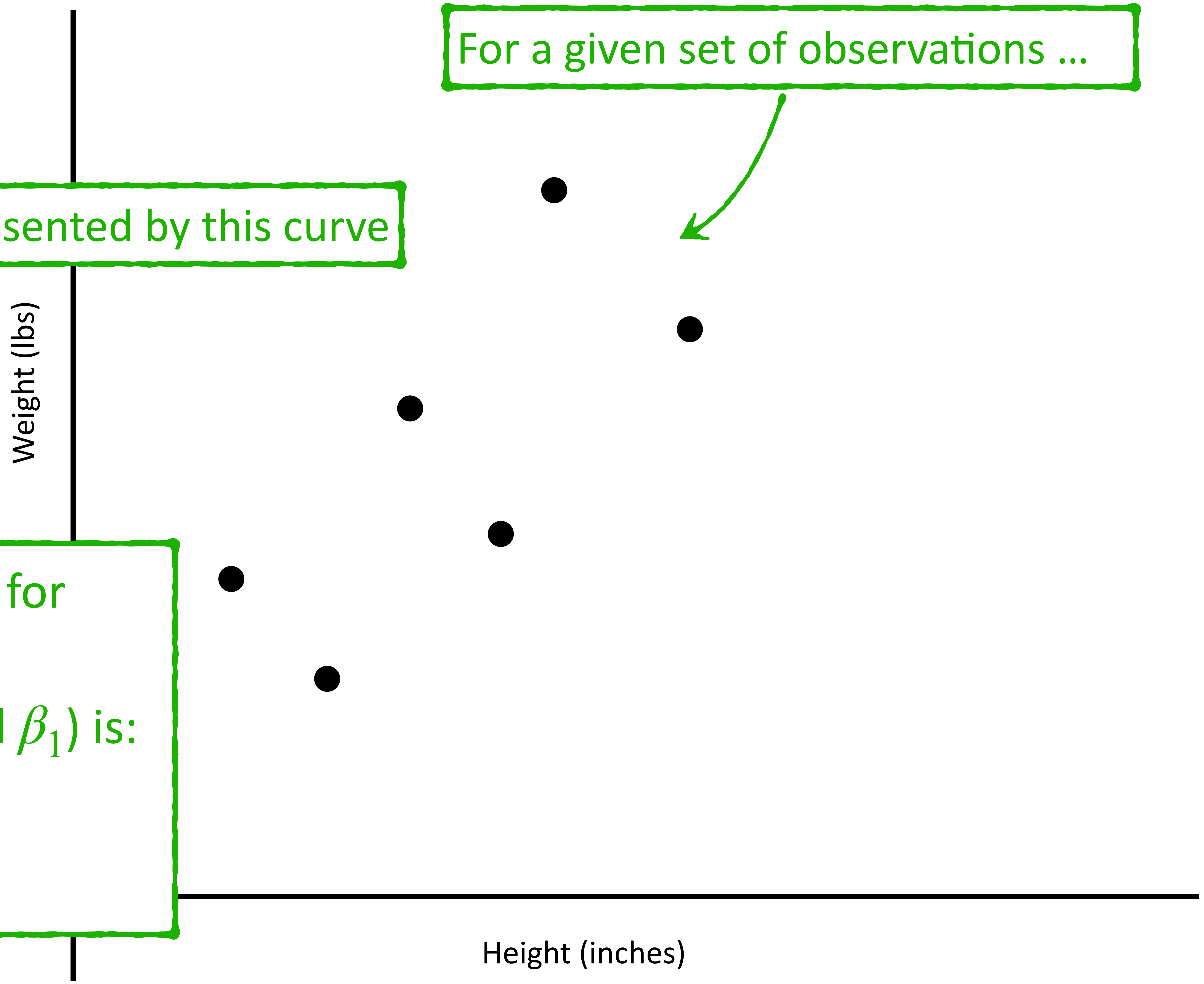
Data is synthetic and not plotted to scale

Mean Squared Error (MSE) for various values of β_0 and β_1



...and is represented by this curve

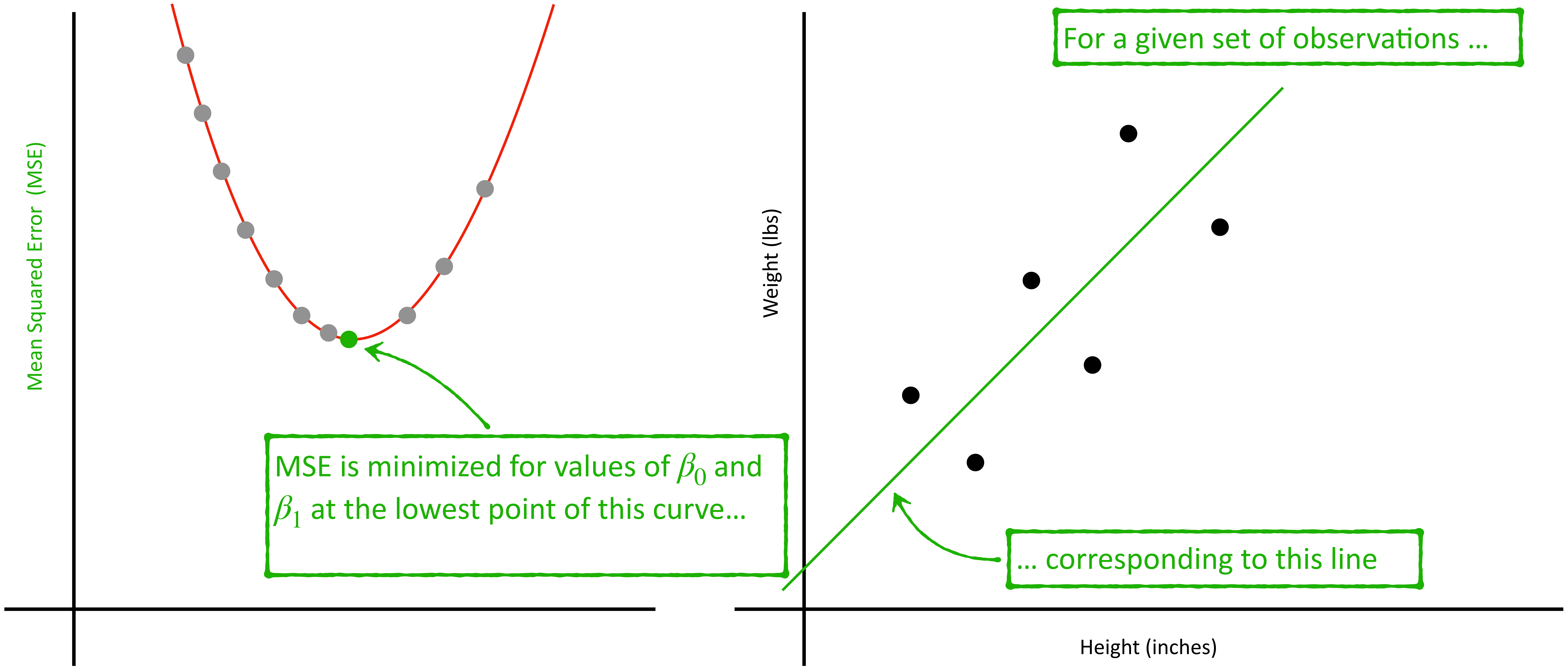
For a given set of observations ...



The **Mean Squared Error (MSE)** for different values of β_0 and β_1 (expressed as function of β_0 and β_1) is:

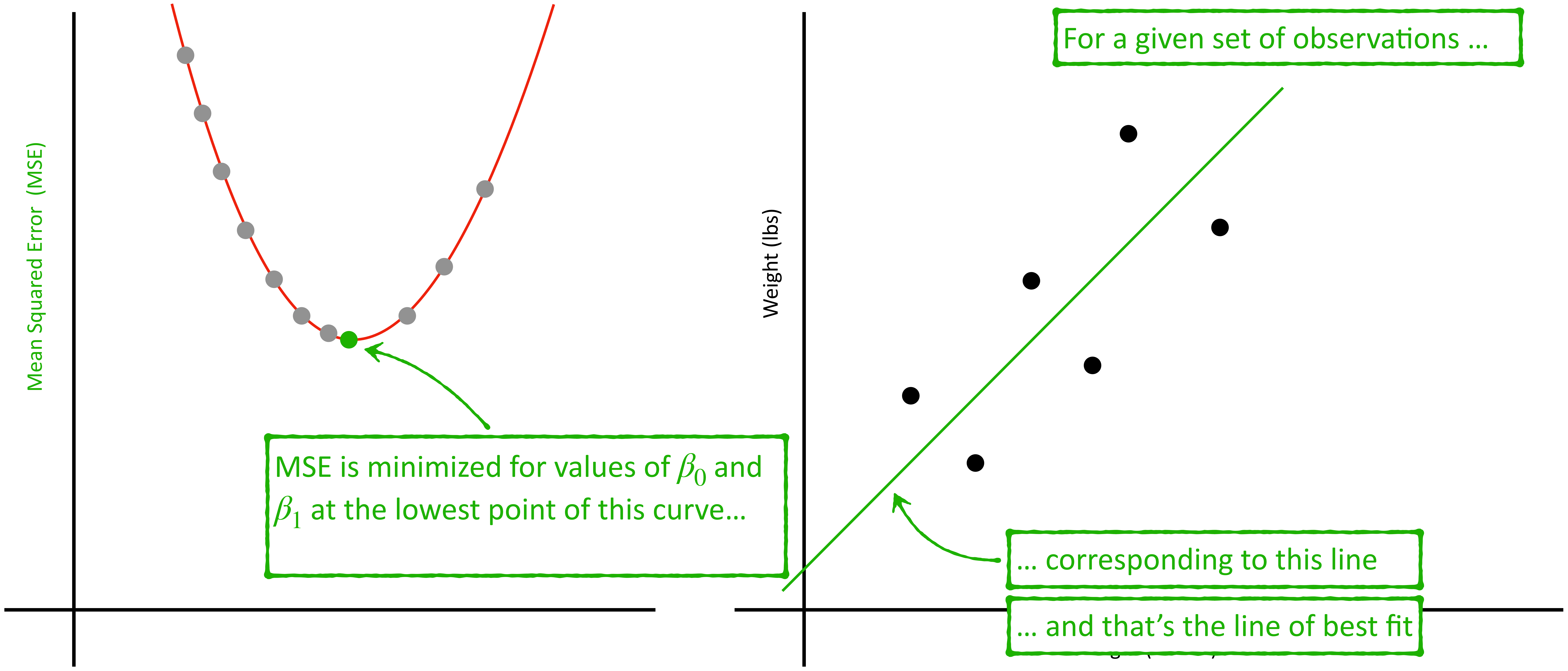
$$f(\beta_0, \beta_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Mean Squared Error (MSE) for various values of β_0 and β_1



Mean Squared Error (MSE) for various values of β_0 and β_1

Simple Linear Regression



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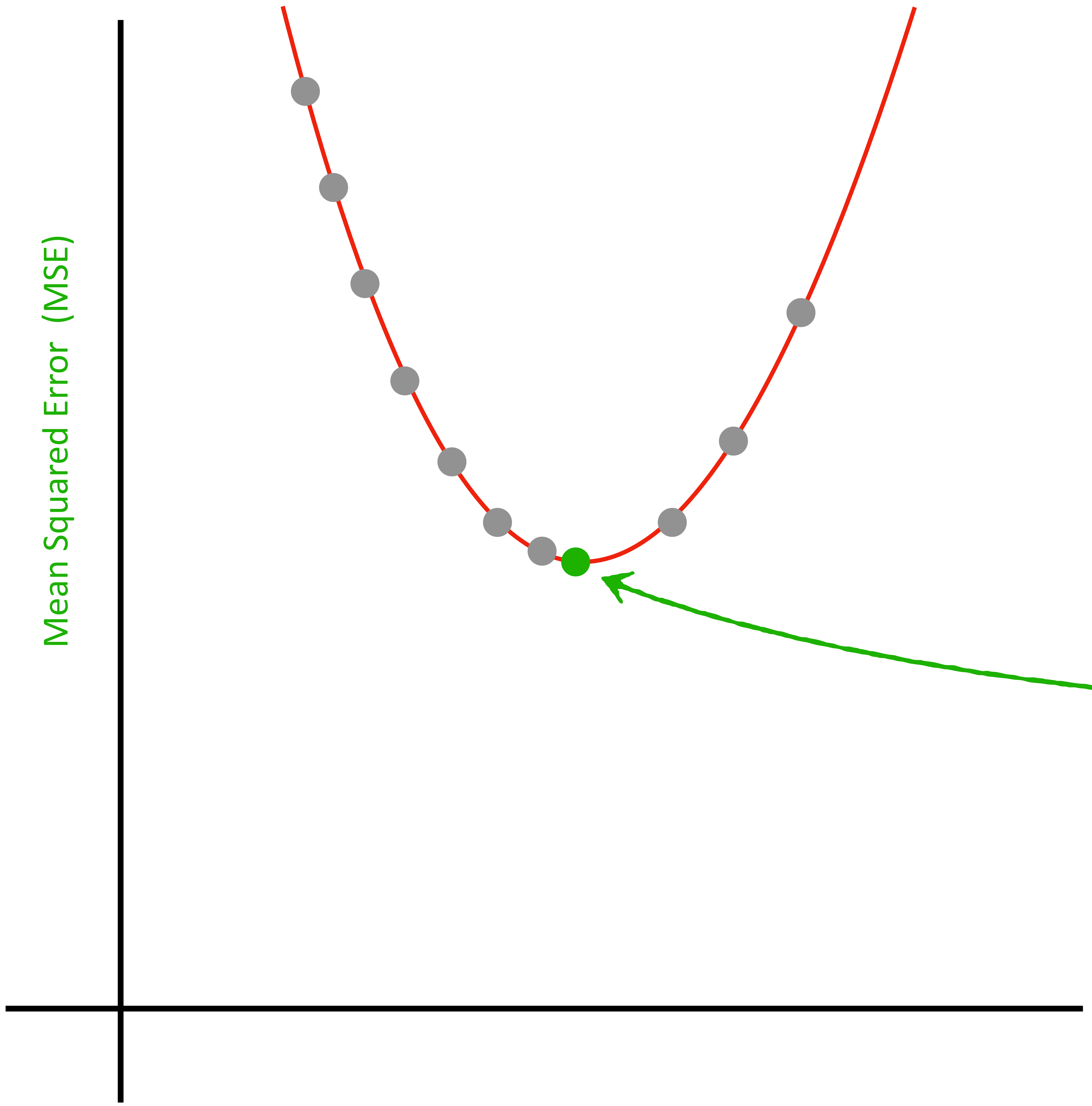

MSE is minimized when the first derivative w.r.t β_0 and β_1 equals 0

See Tutorial on Differential Calculus

Mean Squared Error (MSE) for various values of β_0 and β_1

The **Mean Squared Error (MSE)** is

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

Partial Derivatives w.r.t β_0 andβ_1

$$\frac{\partial}{\partial \beta_0}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 = 0 \quad \cdots \; eq(1)$$

$$\frac{\partial}{\partial \beta_1}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 = 0 \quad \cdots \; eq(2)$$

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Solving both equations for β_0 and β_1 we get…

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Simple Linear Regression

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$$\beta_0 = \frac{\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i}{n}$$

$$\beta_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

This is known as the **Closed Form Solution** for Simple Linear Regression

For the details on how the two equations are solved see [Proof of the Closed Form Solution]

Simple Linear Regression

Mean Squared Error (MSE) for various values of β_0 and β_1

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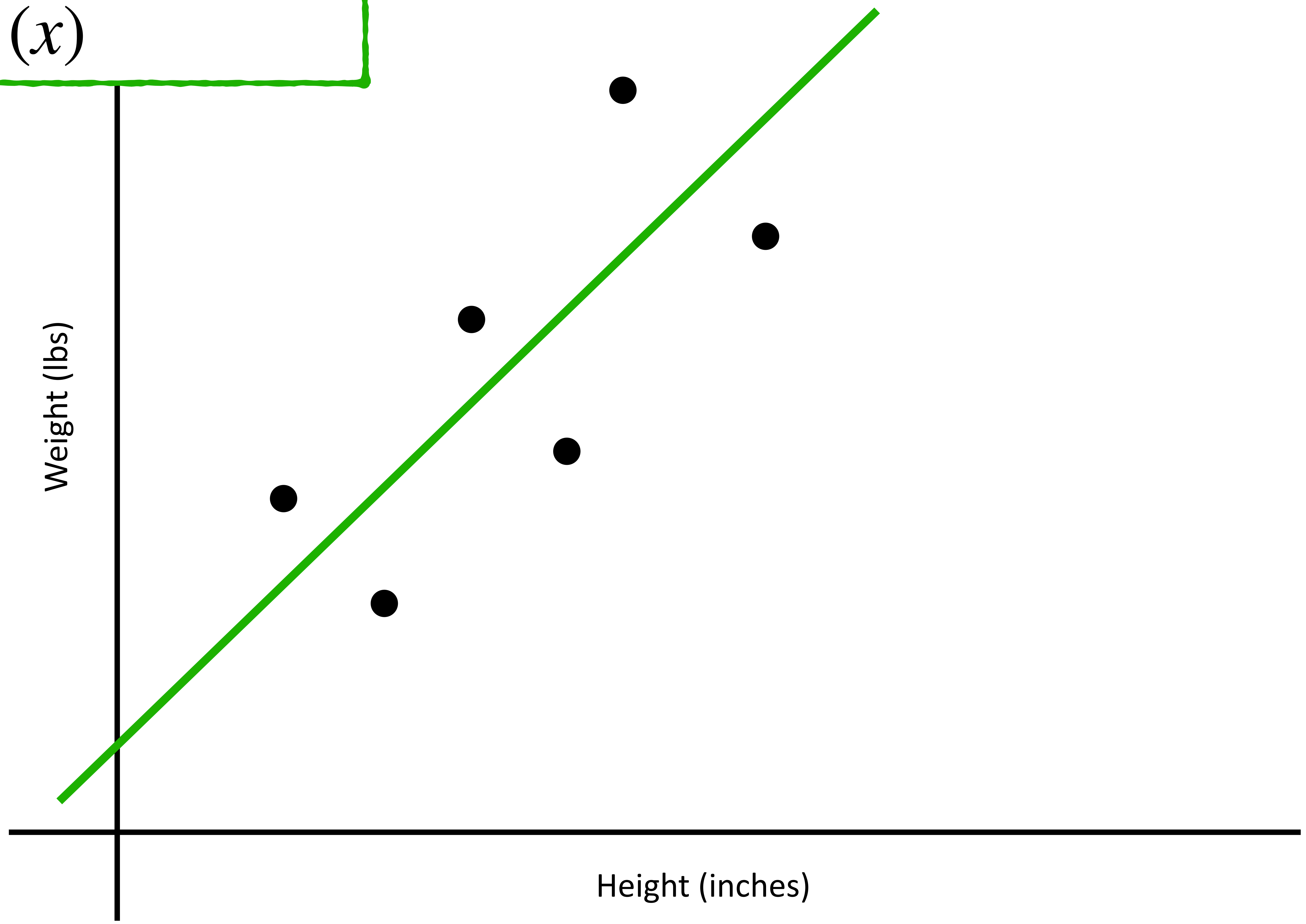
Simple Linear Regression

1 dependent variable

1 independent variable

$$\hat{y} = \beta_0 + \beta_1 x$$

Eg: Predict Weight ($\hat{y}$) of a person given Height (x)



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Simple Linear Regression

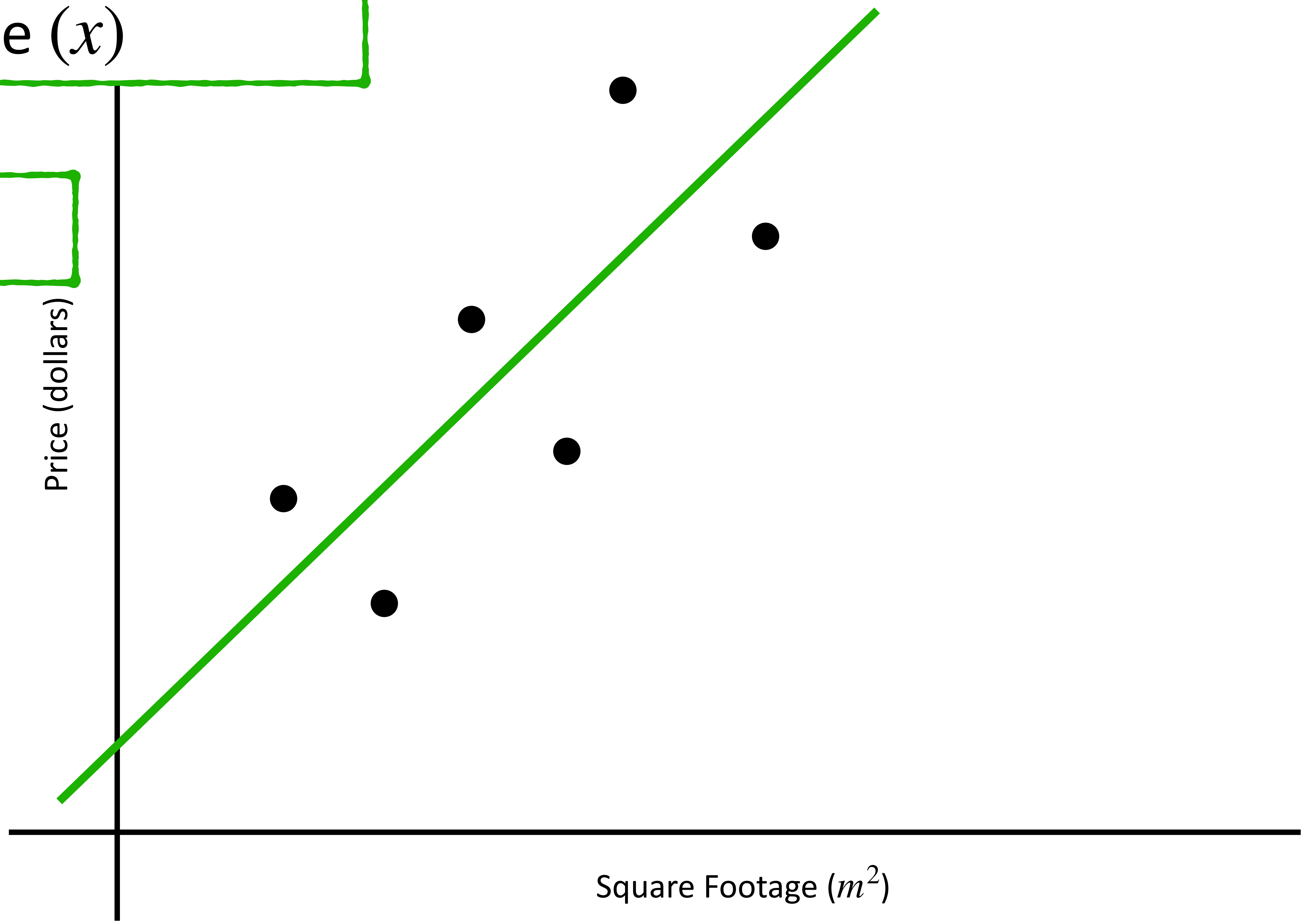
$$\hat{y} = \beta_0 + \beta_1 x$$

1 dependent variable

1 independent variable

Eg: Predict Price of a house ($\hat{y}$) given Square Footage (x)

2 Parameters - β_0 and β_1



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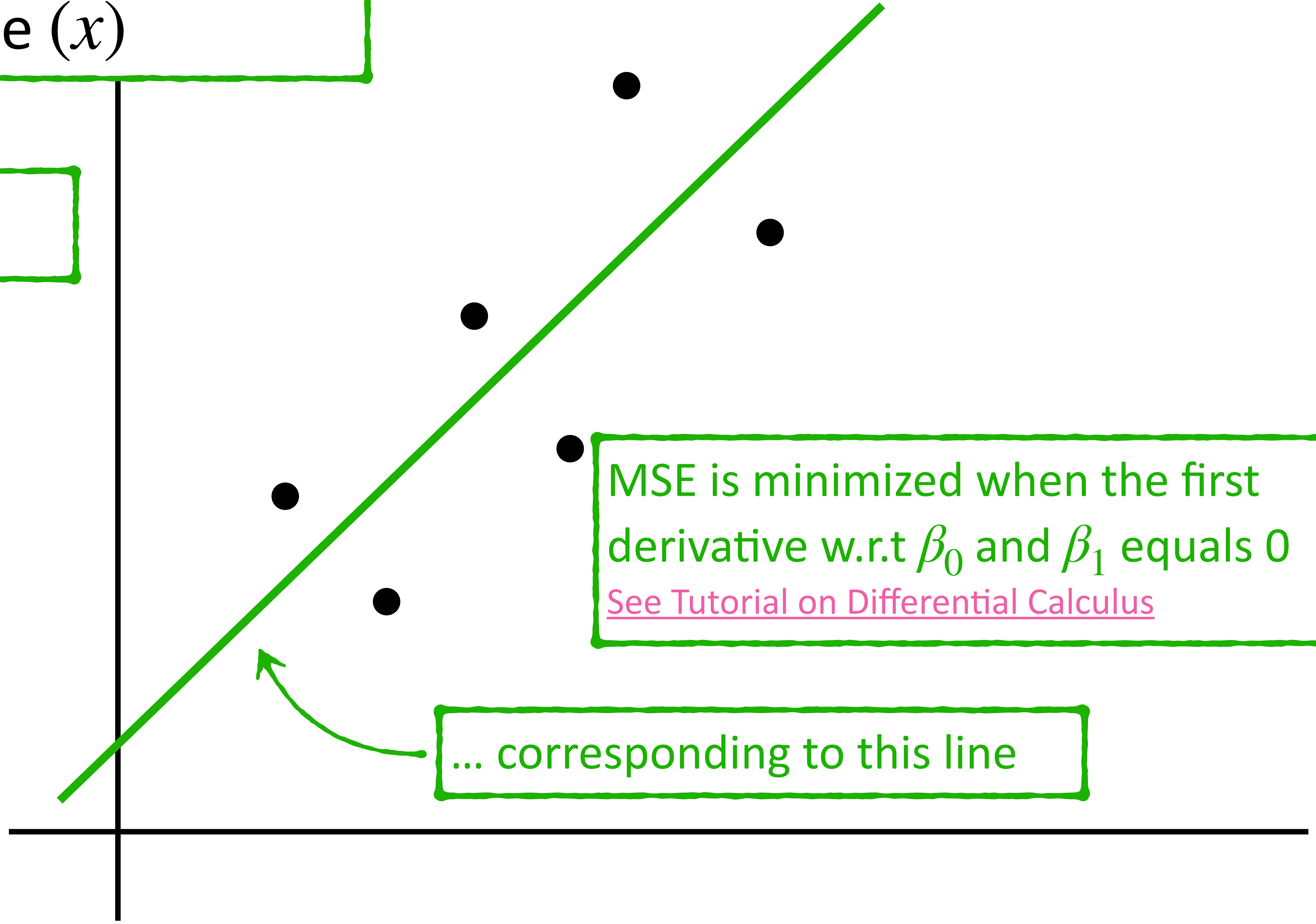
Eg: Predict Price of a house ($\hat{y}$) given Square Footage (x)

2 Parameters - β_0 and β_1

Goal: Find the values of β_0 and β_1 that minimizes the Mean Squared Error (MSE)

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$



MSE is minimized when the first derivative w.r.t β_0 and β_1 equals 0

See Tutorial on Differential Calculus

... corresponding to this line

Data is synthetic and not plotted to scale

Simple Linear Regression

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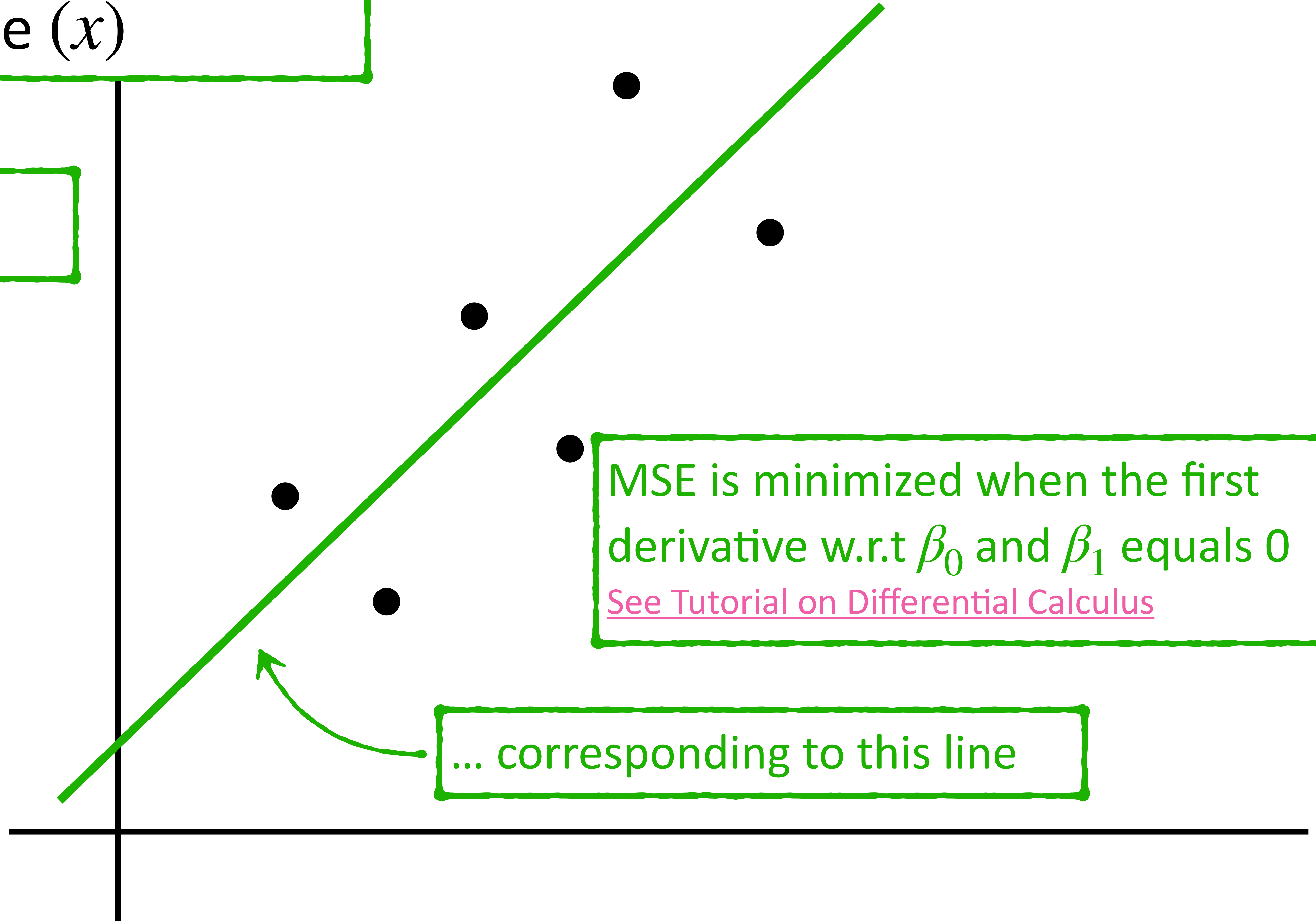

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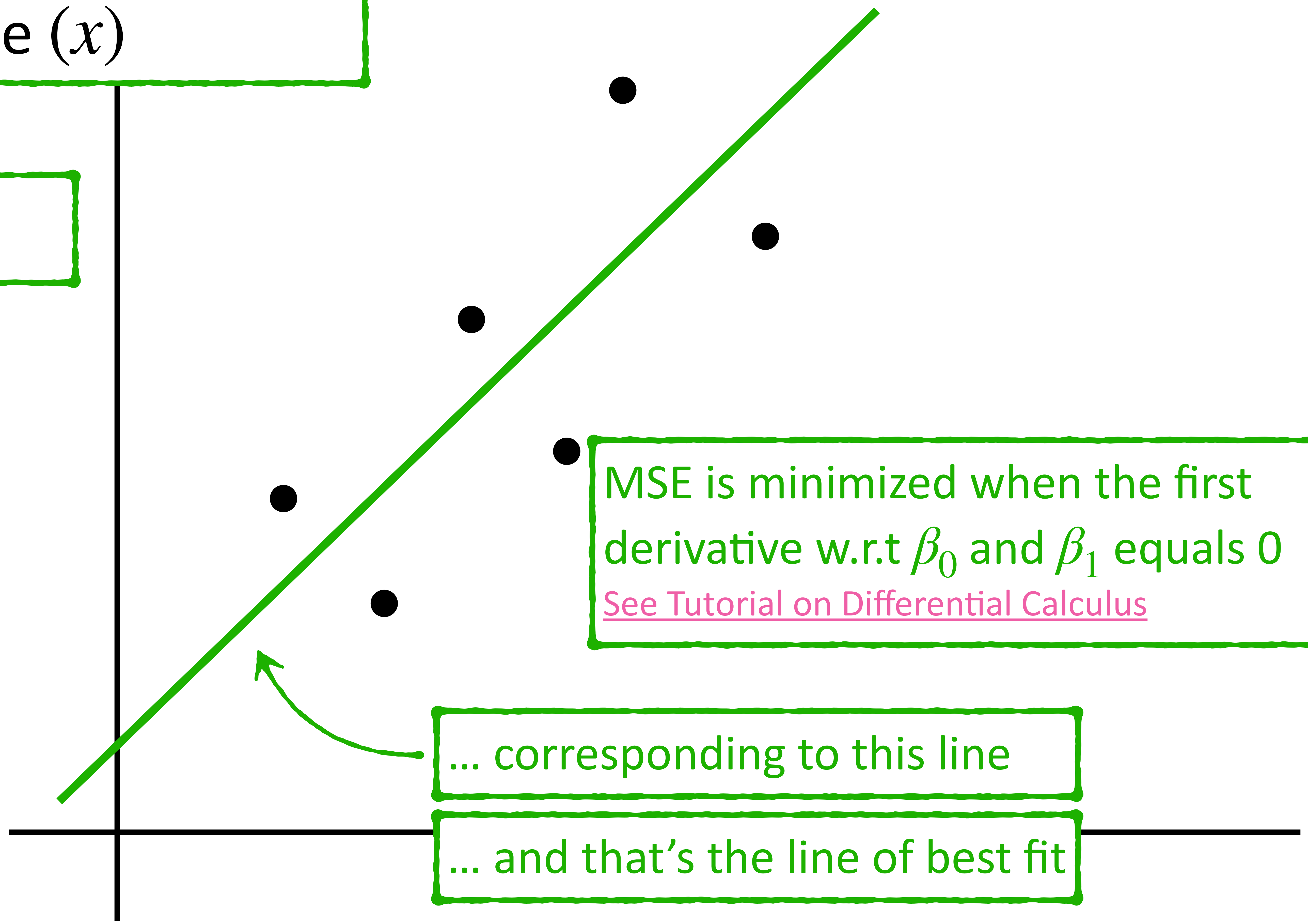

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… corresponding to this line

… and that's the line of best fit

Data is synthetic and not plotted to scale

Simple Linear Regression

1 dependent variable

1 independent variable

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\beta_0 = \frac{\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i}{n}$$

$$\beta_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2}$$

We solve for two Parameters - β_0 and β_1 - by solving two equations…

$$\frac{\partial}{\partial \beta_0} \frac{1}{n} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 = 0 \quad \cdots \quad eq(1)$$

$$\frac{\partial}{\partial \beta_1} \frac{1}{n} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 = 0 \quad \cdots \quad eq(2)$$

MSE is minimized when the first derivative w.r.t β_0 and β_1 equals 0

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Data is synthetic and not plotted to scale

Simple Linear Regression

$$\hat{y} = \beta_0 + \beta_1 x$$

Predict Price $(\hat{y})$ of a house given Square Footage (x)

Simple Linear Regression

- 1 dependent variable $\hat{y}$
- 1 independent variable x
- 2 Parameters - β_0 and β_1
- We solve 2 equations to find the values of β_0 and β_1

Data is synthetic and not plotted to scale

Multiple Regression

What if we wanted to predict price of a house, given Square Footage **and** Number of bedrooms?

Multiple Regression

1 dependent variable $\hat{y}$

2 independent variables

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

x_1 represents the square footage

x_2 represents the number of bedrooms

What if we wanted to predict price of a house, given Square Footage **and** Number of bedrooms?

Data is synthetic and not plotted to scale

Multiple Regression

1 dependent variable $\hat{y}$

2 independent variables

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$\hat{x}_1$ represents the square footage

$\hat{x}_2$ represents the number of bedrooms

Goal: Find the values of β_0, β_1 and β_2 that minimizes the Sum of Squared Residuals (SSR)

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

MSE is minimized when the first derivative w.r.t β_0, β_1 and β_2 equals 0

See Tutorial on Differential Calculus

We solve for three Parameters - β_0, β_1 and β_2 - by solving three equations…

$$\frac{\partial}{\partial \beta_0}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2 = 0$$

$$\frac{\partial}{\partial \beta_1}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2 = 0$$

$$\frac{\partial}{\partial \beta_2}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2 = 0$$

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Multiple Regression

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2 independent variables

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$\hat{x}_1$ represents the square footage

$\hat{x}_2$ represents the number of bedrooms

Goal: Find the values of β_0, β_1 and β_2 that minimizes the Sum of Squared Residuals (SSR)

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2$$

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Data is synthetic and not plotted to scale

Lets generalize this…

Data is synthetic and not plotted to scale

Multiple Regression

A linear model with...

1 dependent variable $\hat{y}$

2 independent variables

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

Has 3 parameters

And a cost function...

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2$$

That can be minimized by solving 3 equations

$$\frac{\partial}{\partial \beta_0}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2 = 0$$

$$\frac{\partial}{\partial \beta_1}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2 = 0$$

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Multiple Regression

A linear model with...

1 dependent variable $\hat{y}$

3 independent variables

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

Has 4 parameters

And a cost function...

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2$$

That can be minimized by solving 4 equations

$$\frac{\partial}{\partial \beta_0}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2 = 0$$

$$\frac{\partial}{\partial \beta_1}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2 = 0$$

$$\frac{\partial}{\partial \beta_2}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2 = 0$$

$$\frac{\partial}{\partial \beta_3}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2 = 0$$

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Multiple Regression

A linear model with...

1 dependent variable $\hat{y}$

$k-1$ independent variables $x_1 \ldots x_{k-1}$

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_{k-1} x_{k-1}$$

Has k parameters

And a cost function...

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} - \ldots - \beta_{k-1} x_{k-1i})^2$$

That can be minimized by solving k equations

$$\frac{\partial}{\partial \beta_0}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} - \ldots - \beta_{k-1} x_{k-1i})^2 = 0$$

$$\frac{\partial}{\partial \beta_1}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} - \ldots - \beta_{k-1} x_{k-1i})^2 = 0$$

$$\frac{\partial}{\partial \beta_2}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} - \ldots - \beta_{k-1} x_{k-1i})^2 = 0$$

$$\vdots$$

$$\frac{\partial}{\partial \beta_{k-1}}\frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} - \ldots - \beta_{k-1} x_{k-1i})^2 = 0$$

k independent variables $x_1 \ldots x_k$

$$\hat{y} = \beta_0 + \beta_1\hat{x}_1 + \beta_2\hat{x}_2 + \beta_3\hat{x}_3 + \ldots + \beta_k\hat{x}_k$$

Has $k + 1$ p

And a cost f

$$\sum_{i=0}^{n}(y_i - \beta_0 - \beta_1\hat{x}_{1i} - \beta_2\hat{x}_{2i} - \beta_3\hat{x}_{3i} - \ldots - \beta_k\hat{x}_{ki})^2$$

That can be minimized by solving $k + 1$ equations

$$- \beta_k\hat{x}_{ki})^2 = 0$$

$$- \beta_k\hat{x}_{ki})^2 = 0$$

$$- \beta_k\hat{x}_{ki})^2 = 0$$

$$\vdots$$

$$\frac{\partial}{\partial\beta_n}\sum_{i=0}^{n}(y_i - \beta_0 - \beta_1\hat{x}_{1i} - \beta_2\hat{x}_{2i} - \beta_3\hat{x}_{3i} - \ldots - \beta_k\hat{x}_{ki})^2 = 0$$

Solving k linear equations isn't practical

Lets use a Matrix

Simple Linear Regression

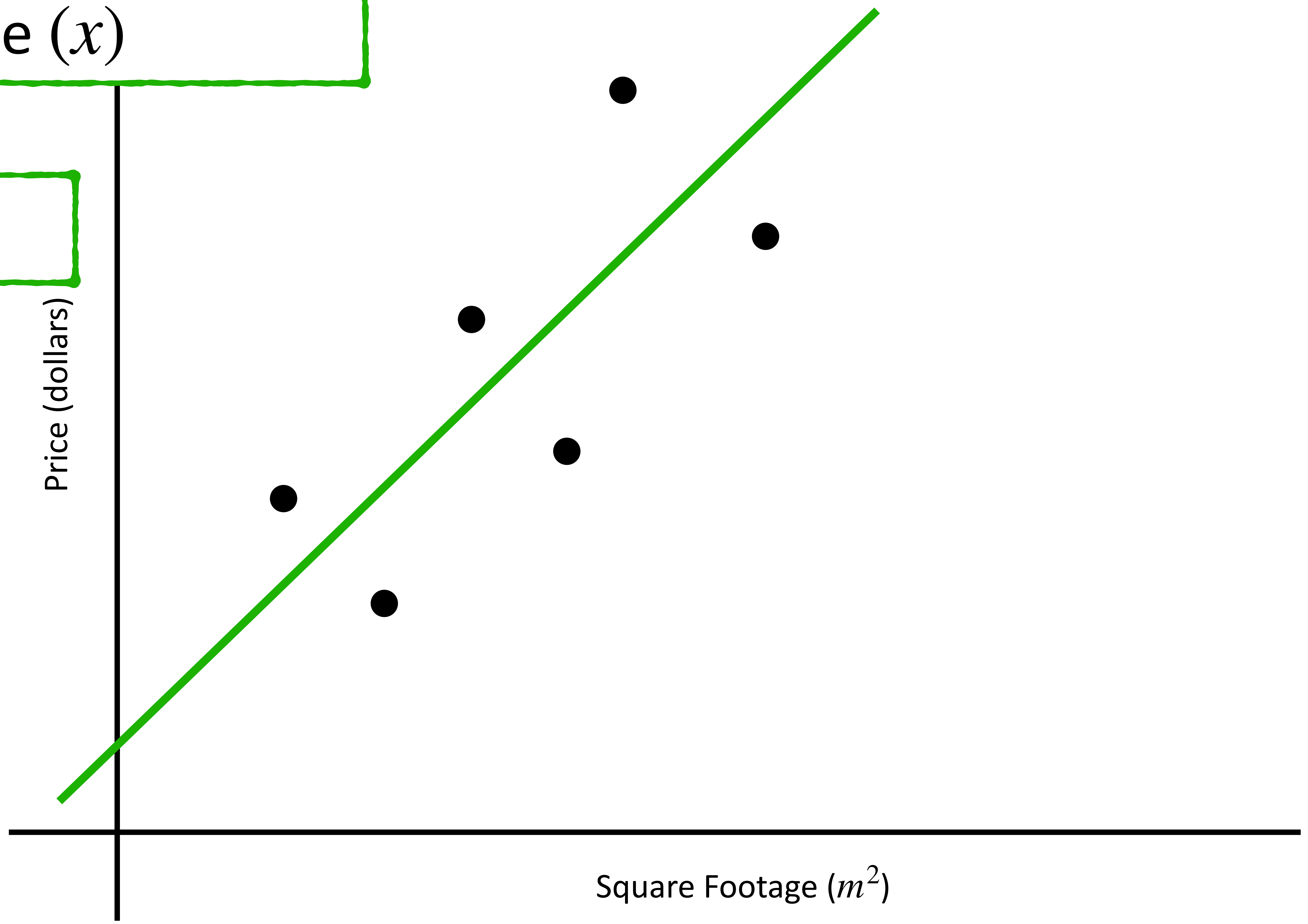
$$\hat{y} = \beta_0 + \beta_1 x$$

1 dependent variable

1 independent variable

Eg: Predict Price of a house ($\hat{y}$) given Square Footage (x)

2 Parameters - β_0 and β_1

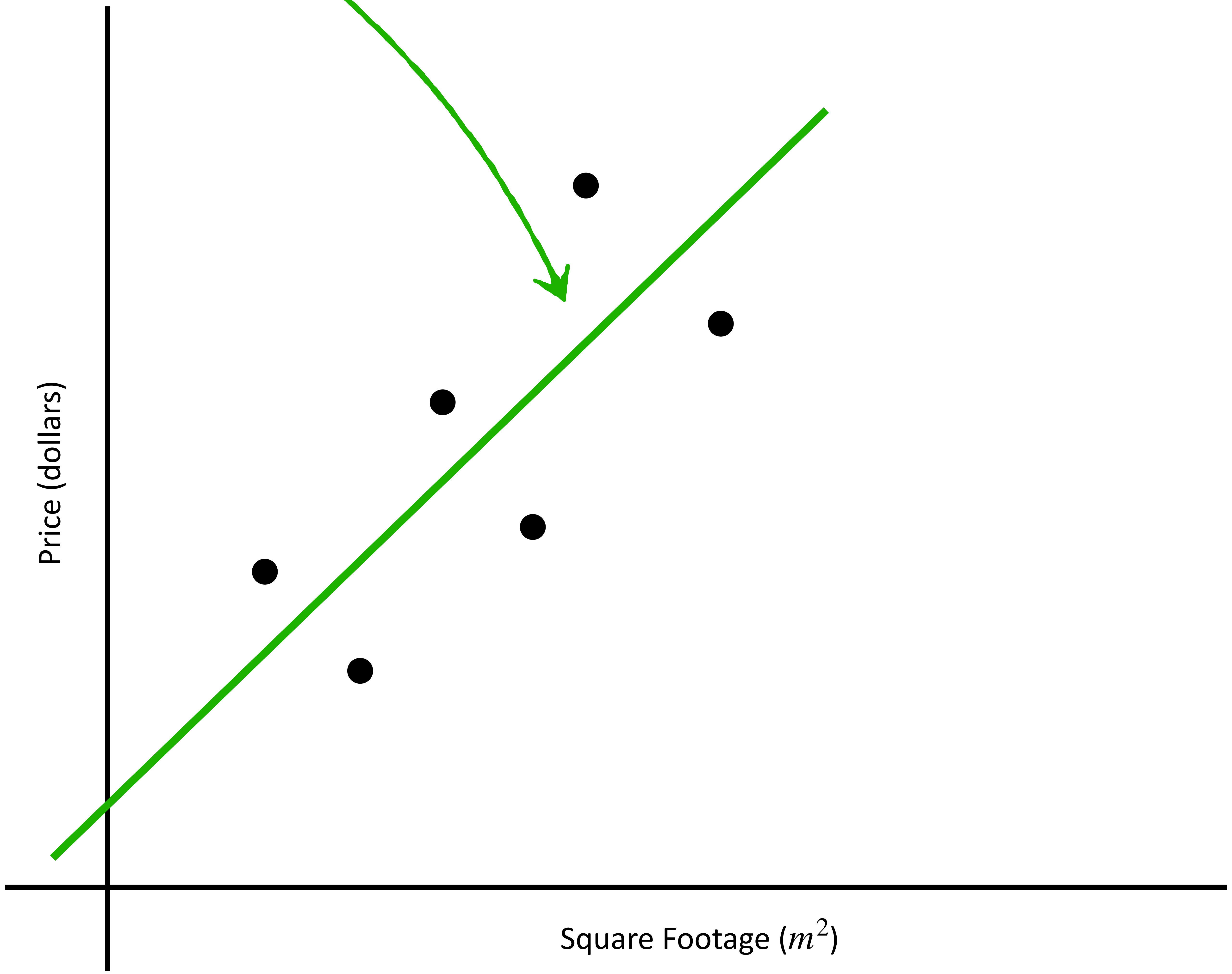


Data is synthetic and not plotted to scale

Simple Linear Regression

Line of Best Fit...

$$\hat{y} = \beta_0 + \beta_1 x$$



Data is synthetic and not plotted to scale

Simple Linear Regression

Line of Best Fit…

…can be represented as a matrix

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\hat{y}_1 = 1 \times \beta_0 + x_1 \times \beta_1$$

$$\hat{y}_2 = 1 \times \beta_0 + x_2 \times \beta_1$$

$$\hat{y}_3 = 1 \times \beta_0 + x_3 \times \beta_1$$

$$\vdots$$

$$\hat{y}_n = 1 \times \beta_0 + x_n \times \beta_1$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

- 1 dependent variable $\hat{y}$
- 1 independent variables x
- 2 parameters - β_0 and β_1

Data is synthetic and not plotted to scale

Simple Linear Regression

Linear Model in 2 Dimensions

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\hat{y}_1 = 1 \times \beta_0 + x_1 \times \beta_1$$

$$\hat{y}_2 = 1 \times \beta_0 + x_2 \times \beta_1$$

$$\hat{y}_3 = 1 \times \beta_0 + x_3 \times \beta_1$$

$$\vdots$$

$$\hat{y}_n = 1 \times \beta_0 + x_n \times \beta_1$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

- 1 dependent variable $\hat{y}$
- 1 independent variables x
- 2 parameters - β_0 and β_1

Data is synthetic and not plotted to scale

Multiple Regression

Linear Model in 3 Dimensions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$\hat{y}_1 = 1 \times \beta_0 + x_{11} \times \beta_1 + x_{21} \times \beta_1$$

$$\hat{y}_2 = 1 \times \beta_0 + x_{12} \times \beta_1 + x_{22} \times \beta_2$$

$$\hat{y}_3 = 1 \times \beta_0 + x_{13} \times \beta_1 + x_{23} \times \beta_2$$

$$\vdots$$

$$\hat{y}_n = 1 \times \beta_0 + x_{1n} \times \beta_1 + x_{2n} \times \beta_2$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ 1 & x_{13} & x_{23} \\ \vdots & & \\ 1 & x_{1n} & x_{2n} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

- 1 dependent variable $\hat{y}$
- 2 independent variables x_1 and x_2
- 3 parameters - β_0, $\beta_1$and β_2

Multiple Regression

Linear Model in 4 Dimensions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

$$\hat{y}_n = 1 \times \beta_0 + x_{1n} \times \beta_1 + \hat{x}_{2n} \times \beta_2 + x_{3n} \times \beta_3$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 1 & x_{12} & x_{22} & x_{32} \\ 1 & x_{13} & x_{23} & x_{33} \\ \vdots & & & \\ 1 & x_{1n} & x_{2n} & x_{3n} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

- 1 dependent variable $\hat{y}$
- 3 independent variables x_1, x_2 and x_3
- 4 parameters - β_0, β_1, β_2 and β_3

Data is synthetic and not plotted to scale

Multiple Regression

Linear Model in k Dimensions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_{k-1} x_{k-1}$$

$$\hat{y}_n = 1 \times \beta_0 + x_{1n} \times \beta_1 + x_{2n} \times \beta_2 + x_{3n} \times \beta_3 + \ldots + x_{k-1n} \times \beta_{k-1}$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} & . & . & . & x_{(k-1)1} \\ 1 & x_{12} & x_{22} & x_{32} & . & . & . & x_{(k-1)2} \\ 1 & x_{13} & x_{23} & x_{33} & . & . & . & x_{(k-1)3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} & x_{3n} & . & . & . & x_{(k-1)n} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{k-1} \end{bmatrix}$$

- 1 dependent variable $\hat{y}$
- $k - 1$ independent variables $x_1, x_2, x_3 \ldots x_{k-1}$
- k parameters - $\beta_0, \beta_1, \beta_2, \beta_3 \ldots \beta_{k-1}$

Data is synthetic and not plotted to scale

Multiple Regression

Linear Model in k Dimensions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_{k-1} x_{k-1}$$

$$\hat{Y} = X\beta$$

$\hat{Y}$ is a column vector and X is a matrix

$$\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} & . & . & . & x_{(k-1)1} \\ 1 & x_{12} & x_{22} & & & & . & x_{(k-1)2} \\ 1 & x_{13} & x_{23} & & & & . & x_{(k-1)3} \\ \vdots & \vdots & \vdots & . & . & . & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} & x_{3n} & . & . & . & x_{(k-1)n} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{bmatrix}$$

$$\underset{n\times 1}{\hat{Y}} = \underset{n\times k}{X}\ \underset{k\times 1}{\beta}$$

- 1 dependent variable $\hat{y}$
- $k-1$ independent variables $x_1, x_2, x_3 \ldots x_{k-1}$
- k parameters - $\beta_0, \beta_1, \beta_2, \beta_3 \ldots \beta_{k-1}$

Data is synthetic and not plotted to scale

$$\hat{y} = \beta_0 + \beta_1 x$$

Given a matrix (Y) of observations

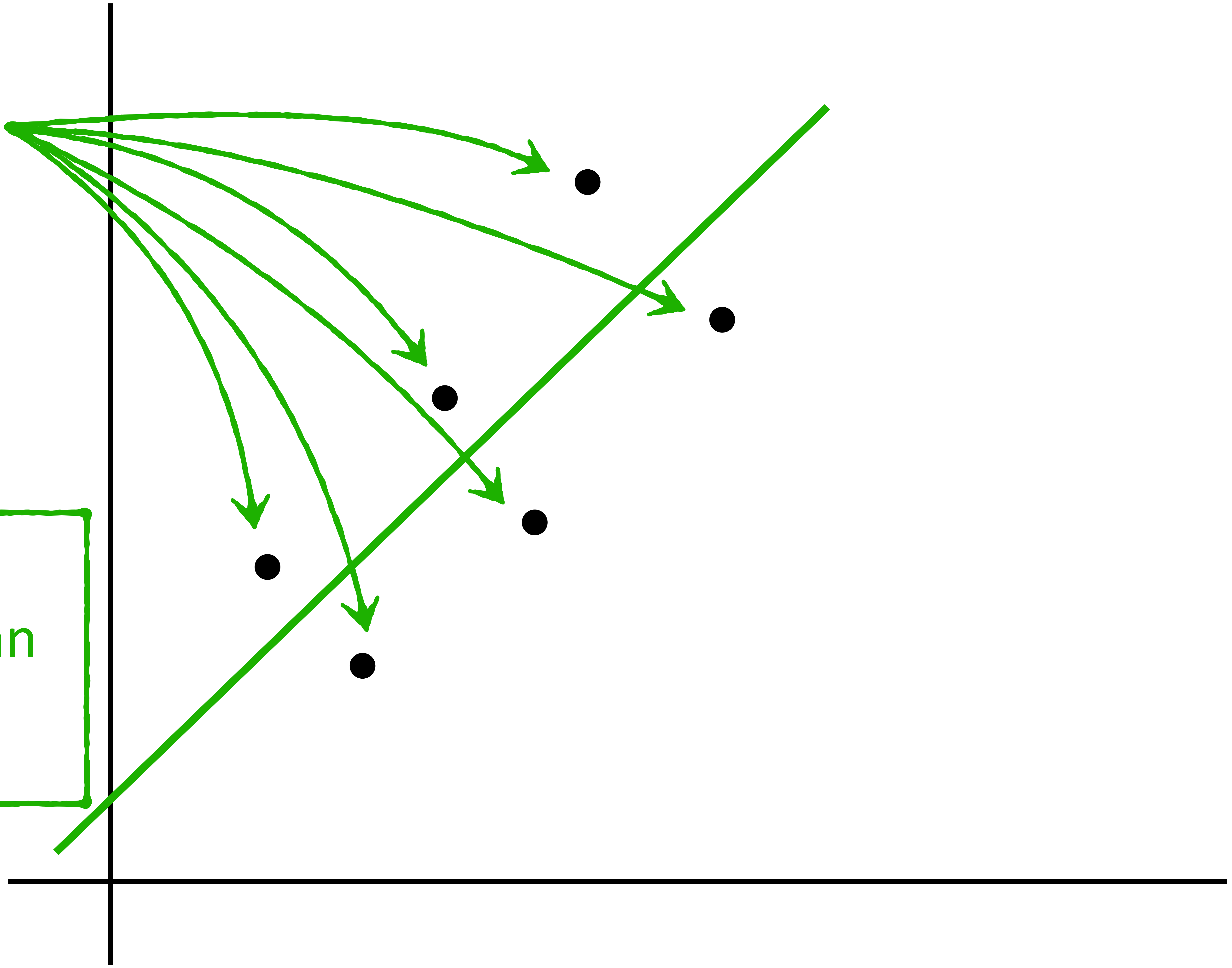
$$\hat{Y} = X\beta$$

The **Mean Squared Error (MSE)**

$$\frac{1}{n} \| Y - \hat{Y} \|^2$$

The two parallel vertical lines mean that this is the Euclidean Norm of the matrix

See Tutorial on Vectors & Matrices



Data is synthetic and not plotted to scale

$$\hat{y} = \beta_0 + \beta_1 x$$

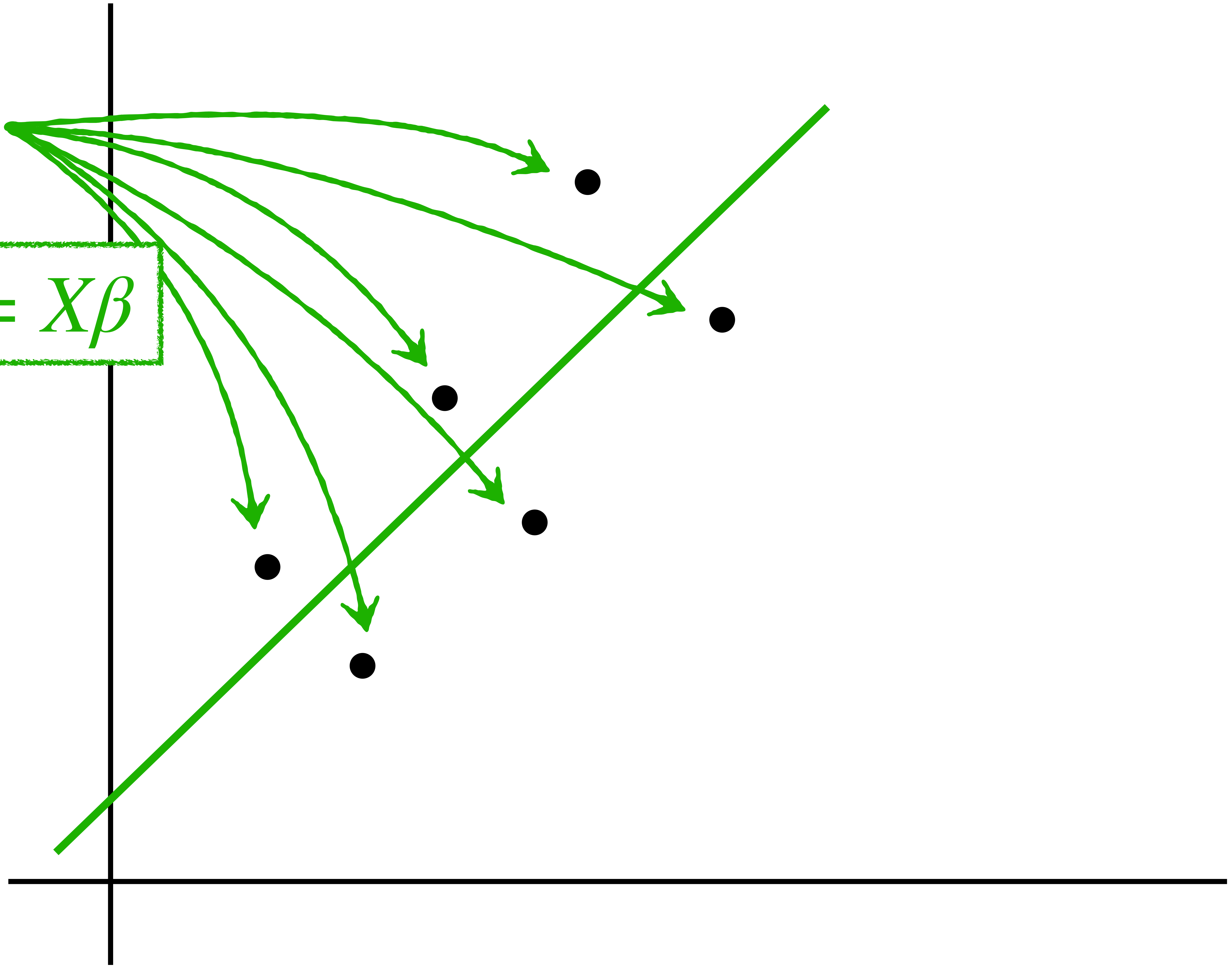
Given a matrix (Y) of observations

$$\hat{Y} = X\beta$$

Substituting $\hat{Y} = X\beta$

The **Mean Squared Error (MSE)**

$$\frac{1}{n} \| Y - \hat{Y} \|^2 = \frac{1}{n} \| Y - X\beta \|^2$$



Data is synthetic and not plotted to scale

Multiple Regression

Linear Model in k Dimensions

$$\hat{Y} = X\beta$$

The **Mean Squared Error (MSE):**

$$\frac{1}{n} \| Y - X\beta \|^2$$

The Problem Statement:

Multiple Regression: Compute the matrix β such that the Mean Squared Error (MSE) is minimized.

Data is synthetic and not plotted to scale

The Problem Statement:

Multiple Regression: Compute the matrix β such that the Mean Squared Error (MSE) is minimized.

$$\frac{1}{n} \| Y - X\beta \|^2$$

This is the cost function (aka loss function) that we must minimize.

$$\frac{\partial}{\partial \beta} \frac{1}{n} \| Y - X\beta \|^2 = 0$$

We take the partial derivate and set it = 0

$$\beta = (X^T X)^{-1} X^T Y$$

Solving for β

For the details of the derivation see the tutorial on [Derivation of the Matrix Form for Multiple Regression]

Data is synthetic and not plotted to scale

The Problem Statement:

Multiple Regression: Compute the matrix β such that the Mean Squared Error (MSE) is minimized.

Solution:

$$\beta = (X^T X)^{-1} X^T Y$$

This is the **Closed form solution** for **Multiple Regression**. However this requires inverting a matrix which is not always possible.

For more details on the reasons why see the tutorial on Vectors & Matrices

Related Tutorials & Textbooks

Simple Linear Regression

A statistical technique of making predictions from data. The tutorial introduces a linear model in two dimensions and uses that model to predict the value of one dependent variable given one independent variable.

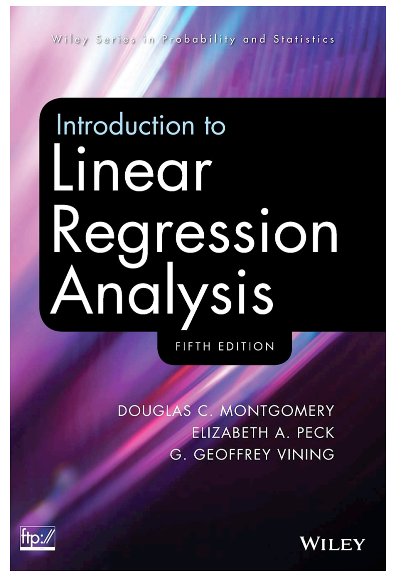
Multiple Regression: Deriving the Matrix Form

A proof of the closed form matrix representation of the multiple regression model. This closed form represents a linear model with k + 1 parameters and solves for the matrix β. This requires a matrix inverse that is not always possible.

Gradient Descent for Multiple Regression

Gradient Descent algorithm for multiple regression and how it can be used to optimize k + 1 parameters for a Linear model in multiple dimensions.

Recommended Textbooks



Introduction to Linear Regression Analysis

by Douglas C. Montgomery, Elizabeth A. Peck, G. Geoffrey Vining

For a complete list of tutorials see:
https://arrsingh.com/ai-tutorials